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SHEAR WAVES IN AN ELASTIC WEDGE

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ABSTRACT

An elastic wedge of interior angle $\varkappa\pi$ is subjected to spatially uniform but time dependent shear tractions, which are applied to one or both faces of the wedge, parallel to the line of intersection of the faces. The transient wave propagation problem is solved by taking advantage of the dynamic similarity which characterizes problems without a fundamental length in the geometry. The shear stress $\tau_{\theta z}$ is evaluated, and it is found that the singularity near the vertex of the wedge is of the form $r^{(1/\varkappa)-1}/(1-\varkappa)$. The results show that the stress is not singular for interior angles less than π. As a special case we obtain the dynamic shear stress generated by the sudden opening of a semi-infinite crack in a homogeneously sheared unbounded medium.

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INTRODUCTION

Problems that are concerned with the propagation of small deformations in linearly elastic solids are generally solved by means of Fourier transform techniques. If the region in space is unbounded, and if no characteristic length of the geometry enters the formulation, it may reasonably well be expected that a closed-form solution can be worked out. In obtaining this solution the use of Fourier transforms becomes, however, less attractive for more complicated regions in space, such as wedges. In this paper we consider, therefore, an alternative method of solution which is based on the dynamic similarity that characterizes problems without a fundamental length. This method, which has been used extensively in supersonic aerodynamics, [1], [2], was applied by Miles to wave propagation problems in homogeneous elastic solids [3].

We consider the transient waves generated by spatially uniform but time dependent shear tractions which are applied to one face of a wedge in a direction parallel to the line of intersection of the faces. The other face is assumed free of surface tractions. Once the solution to this problem has been obtained we can, for arbitrary vertex angles, easily construct the solution for the cases where both faces are subjected to shear tractions, or where one face is clamped. As an interesting special solution we obtain the dynamic shear stress generated by the sudden opening of a semi-infinite shear crack in an unbounded medium. For the general case special attention is devoted to the singularity of the shear stress near the vertex of the wedge.

The transient diffraction of plane waves by a wedge in an acoustic medium was treated earlier by essentially the same method by Miles [4] and Keller and Blank [5] .

GOVERNING EQUATIONS

A homogeneous, isotropic, elastic wedge of interior angle $\varkappa\pi$, see Fig. 1, whose faces are defined by $\theta = 0$ and $\theta = \varkappa\pi$, respectively, is subjected on the face $\theta = 0$ to a uniform but time dependent shear traction $\tau_{\theta z}$. For the time being we assume $\varkappa \geq \frac{1}{2}$; the case $\varkappa < \frac{1}{2}$ will be considered later. The shear traction generates horizontally polarized shear motion in the z-direction only, which is governed by the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \quad , \tag{1}$$

where w is the displacement in the z-direction, $c = (\mu/\rho)^{\frac{1}{2}}$ is the velocity of shear waves, and r, θ, z are cylindrical coordinates. It is assumed that the wedge is at rest prior to $t = 0$

$$t < 0 \qquad w(r,\theta,t) = \dot{w}(r,\theta,t) \equiv 0 \quad . \tag{2}$$

The displacement field generated by a uniform surface traction of arbitrary time dependence can be obtained by Duhamel superposition, once the displacements for a surface traction varying with time as the Dirac delta function have been found. We thus first consider the following boundary conditions

$$\theta = 0\ ,\ r \geq 0: \qquad \tau_{\theta z} = \frac{\mu}{r}\frac{\partial w}{\partial \theta} = \tau_1\,\delta(t) \tag{3}$$

$$\theta = \varkappa\pi\ ,\ r \geq 0: \qquad \tau_{\theta z} = \frac{\mu}{r}\frac{\partial w}{\partial \theta} \equiv 0 \quad . \tag{4}$$

The problem at hand consists of finding a solution of Eq. (1), satisfying the initial conditions (2) and the boundary conditions (3) and (4).

Some observations on the pattern of waves propagating in the wedge can be deduced from elementary principles of wave propagation. The surface traction (3) generates a plane wave with constant displacement

$$w_1 = - \frac{c \tau_1}{\mu} \quad . \tag{5}$$

This wave is called the primary wave, and in Fig. 1 its wavefront at an arbitrary time t is indicated by BD. Since the wedge is at rest prior to time $t = 0$, the medium is undisturbed ahead of the wavefront BD, and as discussed above, the displacement is constant behind it. In addition to the primary wave, the vertex of the wedge, as well as the non-uniformity of the surface traction across the vertex, generates a cylindrical wave with center at O. Since the displacement is continuous across the cylindrical wavefronts we have for $\varkappa \geq \frac{1}{2}$: $w \equiv 0$ along BC.

There is no characteristic length in the problem, and it is thus to be expected that the solution shows dynamic similarity, i.e., the displacement is a function of r/t and θ. It is then expedient to introduce as a new independent variable

$$s = r/t \quad . \tag{6}$$

As a function of s and θ the displacement $w(s,\theta)$ satisfies

$$s^2\left(1 - \frac{s^2}{c^2}\right)\frac{\partial^2 w}{\partial s^2} + s\left(1 - \frac{2s^2}{c^2}\right)\frac{\partial w}{\partial s} + \frac{\partial^2 w}{\partial \theta^2} = 0 \quad . \tag{7}$$

For $s < c$, Eq. (7) is elliptic. Upon introducing Chaplygin's transformation

$$\beta = -\cosh^{-1}\left(\frac{c}{s}\right) , \tag{8}$$

Eq. (7) reduces to

$$\frac{\partial^2 w}{\partial \beta^2} + \frac{\partial^2 w}{\partial \theta^2} = 0 . \tag{9}$$

The solution of Laplace's equation may be written as the real part of an analytic function $\chi(\beta, \theta)$,

$$w(\beta, \theta) = \mathrm{Re}\, \chi(\beta, \theta) . \tag{10}$$

For $s > c$, Eq. (7) is hyperbolic, and may be reduced to the canonical form by the transformation $s = c \sec \alpha$.

From Fig. 1 the region in which Eq. (7) is elliptic is now identified as the cylindrical region ABC. All that now remains to be done is to find a harmonic function $w(\beta, \theta)$ in the segment $0 \le \theta \le \varkappa\pi$, $s \le c$ satisfying boundary conditions which for $\varkappa \ge \frac{1}{2}$ take the form

$$\theta = 0 ,\quad s \le c : \frac{\partial w}{\partial \theta} = 0 \tag{11}$$

$$s = c ,\quad 0 \le \theta \le \frac{\pi}{2} : w = w_1 = -\frac{c\,\tau_1}{\mu} \tag{12}$$

$$s = c ,\quad \frac{\pi}{2} \le \theta \le \varkappa\pi : w = 0 \tag{13}$$

$$\theta = \varkappa\pi ,\quad s \le c : \frac{\partial w}{\partial \theta} = 0 . \tag{14}$$

In Eq. (11) we have used that for $t > 0$ the surface traction has returned to zero, see Eq. (3). The function $w(\beta, \theta)$ can be obtained in several ways; here

we elect to map the segment $s \le c$, $0 \le \theta \le \varkappa\pi$ on the half-plane $\eta \ge 0$ by a conformal mapping which was introduced by Craggs [2]

$$\zeta = \xi + i\eta = \operatorname{sech}\left[(\beta + i\theta)/\varkappa\right] \quad . \tag{15}$$

Equation (15) can also be written as

$$\zeta = \left[\cosh\frac{\beta}{\varkappa}\cos\frac{\theta}{\varkappa} + i\sinh\frac{\beta}{\varkappa}\sin\frac{\theta}{\varkappa}\right]^{-1} \quad . \tag{16}$$

The mapping of the segment $s \le c$, $0 \le \theta \le \varkappa\pi$ on the ζ plane is shown in Fig. 2, where the positions of the various points are indicated. The boundary conditions (11)-(14) are converted into conditions on the real axis, and we find for $\varkappa \ge 1$

$$|\xi| \le 1 \quad : \quad \frac{\partial w}{\partial \eta} = 0 \tag{17}$$

$$1 \le \xi \le 1/\cos\frac{\pi}{2\varkappa} \quad : \quad w = w_1 \tag{18}$$

$$\xi \ge 1/\cos\frac{\pi}{2\varkappa} \quad : \quad w = 0 \tag{19}$$

$$\xi \le -1 \quad : \quad w = 0 \quad . \tag{20}$$

For $1 \ge \varkappa \ge \frac{1}{2}$ the point B' is located on the negative real axis, and (17)-(20) must be modified accordingly.

In this paper we are interested in the stresses, in particular in the shear stress $\tau_{\theta z}$, which may be written in the form

$$\tau_{\theta z} = \frac{\mu}{r}\operatorname{Re}\left[\frac{d\chi}{d\zeta}\,\frac{\partial\zeta}{\partial\theta}\right] \quad . \tag{21}$$

According to (17) $d\chi/d\zeta$ is real for $|\xi| \le 1$, and from (18)-(20) we conclude that $d\chi/d\zeta$ is imaginary elsewhere along the real axis, except at the point

$$\xi = \xi_B = 1/\cos\frac{\pi}{2\varkappa} \quad . \tag{22}$$

At this point Re χ is discontinuous, and Re $d\chi/d\xi$ is a delta function, which implies that $d\chi/d\zeta$ has a simple pole at $\xi = \xi_B$. An expression satisfying the foregoing requirements is

$$\frac{d\chi}{d\zeta} = \frac{i}{(\zeta^2-1)^{\frac{1}{2}}} \frac{B}{\zeta - \xi_B} \quad . \tag{23}$$

The constant B is found by integrating counterclockwise along a small semi-circle around $\zeta = \xi_B$, and equating the results to $-w_1$. We obtain

$$B = \frac{w_1}{\pi} (\xi_B^2 - 1)^{\frac{1}{2}} = -\frac{c\tau_1}{\pi\mu} \tan\frac{\pi}{2\varkappa} \quad . \tag{24}$$

From Eq. (15) we compute

$$\frac{\partial\zeta}{\partial\theta} = -\frac{i}{\varkappa} \frac{\tanh\left(\frac{\beta}{\varkappa} + i\frac{\theta}{\varkappa}\right)}{\cosh\left(\frac{\beta}{\varkappa} + i\frac{\theta}{\varkappa}\right)} \quad . \tag{25}$$

Upon substitution of (16) and (24) into (23), and subsequent substitution of (23) and (25) into (21) we obtain

$$\tau_{\theta z} = \frac{c\tau_1}{r} \frac{\sin\left(\frac{\pi}{2\varkappa}\right)}{\pi\varkappa} \text{Re}\left\{i/\left[\cos\frac{\pi}{2\varkappa} - \cosh\left(\frac{\beta}{\varkappa} + i\frac{\theta}{\varkappa}\right)\right]\right\} \quad . \tag{26}$$

The real part can easily be evaluated, and the result is

$$\tau_{\theta z} = -\frac{c\tau_1}{r} \frac{\sin\left(\frac{\pi}{2\varkappa}\right)}{\pi\varkappa} F_1\left(\frac{r}{ct}, \theta\right) \quad , \tag{27}$$

where

$$F_1\left(\frac{r}{ct}, \theta\right) = \frac{\sinh\frac{\beta}{\varkappa}\sin\frac{\theta}{\varkappa}}{\left(\cos\frac{\pi}{2\varkappa} - \cosh\frac{\beta}{\varkappa}\cos\frac{\theta}{\varkappa}\right)^2 + \left(\sinh\frac{\beta}{\varkappa}\sin\frac{\theta}{\varkappa}\right)^2} \quad . \tag{28}$$

Equation (27) represents the stress within the cylindrical wavefront, $r \le ct$. For $r \ge ct$ and $0 \le \theta \le \frac{\pi}{2}$ we have

$$\tau_{r\theta} = \tau_1 \cos\theta\, \delta(t - \frac{r}{c}\sin\theta) \quad . \tag{29}$$

It is of particular interest to investigate for $t > 0$ the singularity of $\tau_{\theta z}$ as $r \to 0$. To this end we evaluate asymptotic expressions for $\sinh(\beta/\varkappa)$ and $\cosh(\beta/\varkappa)$ for small $\frac{r}{ct}$. Using a well-known representation for $\cosh^{-1}$, we find from (8) and (6):

$$\beta = -\ln\left\{\frac{ct}{r} + \left[\left(\frac{ct}{r}\right)^2 - 1\right]^{\frac{1}{2}}\right\} \quad . \tag{30}$$

Equation (30) is subsequently used to write

$$\cosh\frac{\beta}{\varkappa} = \frac{1}{2}\left[\left(\frac{r}{ct}\right)^{\frac{1}{\varkappa}}\left\{1 + \left[1 - \left(\frac{r}{ct}\right)^2\right]^{\frac{1}{2}}\right\}^{-\frac{1}{\varkappa}} + \left(\frac{ct}{r}\right)^{\frac{1}{\varkappa}}\left\{1 + \left[1 - \left(\frac{r}{ct}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{\varkappa}}\right] \tag{31}$$

$$\sinh\frac{\beta}{\varkappa} = \frac{1}{2}\left[\left(\frac{r}{ct}\right)^{\frac{1}{\varkappa}}\left\{1 + \left[1 - \left(\frac{r}{ct}\right)^2\right]^{\frac{1}{2}}\right\}^{-\frac{1}{\varkappa}} - \left(\frac{ct}{r}\right)^{\frac{1}{\varkappa}}\left\{1 + \left[1 - \left(\frac{r}{ct}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{\varkappa}}\right] \quad . \tag{32}$$

Thus for $\left(\frac{r}{ct}\right) << 1$

$$\cosh\frac{\beta}{\varkappa} \sim \left(\frac{ct}{r}\right)^{\frac{1}{\varkappa}} 2^{\frac{1}{\varkappa} - 1} \tag{33}$$

$$\sinh\frac{\beta}{\varkappa} \sim -\left(\frac{ct}{r}\right)^{\frac{1}{\varkappa}} 2^{\frac{1}{\varkappa} - 1} \quad . \tag{34}$$

For $\varkappa \ge \frac{1}{2}$ the singularity in the shear stress is thus obtained as

$$\tau_{\theta z} \sim \frac{2c\tau_1}{\pi\varkappa}(2ct)^{-\frac{1}{\varkappa}} \sin\left(\frac{\pi}{2\varkappa}\right)\sin\left(\frac{\theta}{\varkappa}\right) r^{\frac{1}{\varkappa} - 1} \quad . \tag{35}$$

We note that the singularity vanishes for $\theta = 0$ and $\theta = \chi\pi$, and reaches a maximum for $\theta = \chi\pi/2$. The singularity disappears altogether if $\chi \leq 1$. The shear stress is thus singular only if the interior angle of the wedge exceeds π.

BOTH FACES SUBJECTED TO SHEAR TRACTIONS

If the faces defined by $\theta = 0$ and $\theta = \kappa\pi$ are subjected to shear tractions $\tau_{\theta z} = \tau_1\,\delta(t)$ and $\tau_{\theta z} = \tau_2\,\delta(t)$, respectively, the shear stress $\tau_{\theta z}$ is obtained by simple superposition. We obtain for $r \le ct$

$$\tau_{\theta z} = -\frac{c}{r}\,\frac{\sin\left(\frac{\pi}{2\kappa}\right)}{\pi\,\kappa}\left\{\tau_1\,F_1\left(\frac{r}{ct},\theta\right) + \tau_2 F_2\left(\frac{r}{ct},\theta\right)\right\} \quad , \tag{36}$$

where $F_1(r/ct, \theta)$ is defined by (28), and

$$F_2\left(\frac{r}{ct},\theta\right) = \frac{\sinh\frac{\beta}{\kappa}\,\sin\frac{\theta}{\kappa}}{\left(\cos\frac{\pi}{2\kappa} + \cosh\frac{\beta}{\kappa}\cos\frac{\theta}{\kappa}\right)^2 + \left(\sinh\frac{\beta}{\kappa}\sin\frac{\theta}{\kappa}\right)^2} \quad . \tag{37}$$

In addition, we have for $r \ge ct$ and $0 \le \theta \le \pi/2$ the plane wave (29), and for $\kappa\pi \ge \theta \ge \kappa\pi - \frac{\pi}{2}$ we have

$$\tau_{\theta z} = \tau_2\cos(\kappa\pi - \theta)\,\delta\left[t - \frac{r}{c}\sin(\kappa\pi - \theta)\right] \quad . \tag{38}$$

It is clear that for this case the singularity is of the form

$$\tau_{\theta z} \sim \frac{2c(\tau_1 + \tau_2)}{\pi\,\kappa}\,(2ct)^{-\frac{1}{\kappa}}\,\sin\left(\frac{\pi}{2\kappa}\right)\sin\left(\frac{\theta}{\kappa}\right)\,r^{\frac{1}{\kappa}-1} \quad . \tag{39}$$

Thus, the singularity vanishes if the two shear tractions are of the same magnitude and sense, i.e., if $\tau_1 = -\tau_2$.

THE SHEAR STRESS FOR STEP SURFACE TRACTIONS

It is of considerable physical interest to compute the dynamic stresses for the case that the surface traction varies as a Heaviside step function,

$$\theta = 0\,, \quad r \geq 0 \; : \quad \tau_{\theta z} = T_1 \, 1(t) \tag{40}$$

$$\theta = \varkappa\pi\,, \; r \geq 0 \; : \quad \tau_{\theta z} = 0 \quad . \tag{41}$$

For $0 \leq \quad \leq \frac{\pi}{2}$ the shear stress $\tau_{\theta z}$ is now obtained as the sum of the integrals of (27) and (29)

$$\tau_{\theta z} = T_1 \cos\theta \, 1(t - \frac{r}{c}\sin\theta) - \frac{c\,T_1}{r} \, \frac{\sin(\frac{\pi}{2\varkappa})}{\pi\varkappa} \, 1(t - \frac{r}{c}) \int_{\frac{r}{c}}^{t} F_1(\frac{r}{cv}, \theta)\, dv \quad . \tag{42}$$

For $\theta \geq \frac{\pi}{2}$ we obtain

$$\tau_{\theta z} = - \frac{c\,T_1}{r} \, \frac{\sin(\frac{\pi}{2\varkappa})}{\pi\varkappa} \, 1(t - \frac{r}{c}) \int_{\frac{r}{c}}^{t} F_1(\frac{r}{cv}, \theta)\, dv \quad . \tag{43}$$

The behavior of $\tau_{\theta z}$ for small values of r is found by introducing the new variable

$$p = \frac{cv}{r} \quad . \tag{44}$$

The integrals in (42) and (43) then become

$$\tau_{\theta z} = - T_1 \, \frac{\sin(\frac{\pi}{2\varkappa})}{\pi\varkappa} \, 1(\frac{ct}{r} - 1) \int_{1}^{ct/r} F_1(p, \theta)\, dp \quad . \tag{45}$$

We are interested in approximating (45) for large values of ct/r . For large p the $\cosh[\beta(p)/\varkappa]$ and $\sinh[\beta(p)/\varkappa]$ may be approximated by, see (31) and (32),

$$\cosh\frac{\beta}{\varkappa} \sim p^{\frac{1}{\varkappa}}\, 2^{\frac{1}{\varkappa}-1} \tag{46}$$

$$\sinh\frac{\beta}{\varkappa} \sim -\, p^{\frac{1}{\varkappa}}\, 2^{\frac{1}{\varkappa}-1} \quad . \tag{47}$$

The integral can now be evaluated, and for $\varkappa \geq \frac{1}{2}$, $\varkappa \neq 1$, and $r/ct << 1$ we find

$$\tau_{\theta z} \sim \frac{T_1}{\pi(\varkappa-1)}\,(2ct)^{-\frac{1}{\varkappa}+1}\, \sin\left(\frac{\pi}{2\varkappa}\right) \sin\left(\frac{\theta}{\varkappa}\right)\, r^{\frac{1}{\varkappa}-1} \quad . \tag{48}$$

For $\varkappa = 1$ and $r/ct << 1$ we obtain

$$\tau_{\theta z} \sim \frac{1}{\pi}\, T_1 \sin\theta\, \ell n\left(\frac{ct}{r}\right) \quad . \tag{49}$$

We note that no singularity occurs if the interior angle of the wedge is less than π .

SPECIAL CASES

The integrals in (42) and (43) can be evaluated with particular ease for three special values of $\varkappa$: $\varkappa = 0.5$, $\varkappa = 1$ and $\varkappa = 2$.

For $\underline{\varkappa = 0.5}$ we have a quarter-space subjected to uniform surface tractions. From Eqs. (27) and (42) it is immediately noted that the cylindrical wave vanishes, as it should, and we thus are just left with the plane primary wave.

The case $\underline{\varkappa = 1}$ is concerned with a half-space. We assume uniform surface tractions that are of different magnitudes for $\theta = 0$ and $\theta = \pi$, respectively. Equation (36) then reduces to

$$\tau_{\theta z} = -\frac{c}{r}\frac{1}{\pi}\frac{(\tau_1 + \tau_2)\sinh\beta\sin\theta}{\cosh^2\beta - \sin^2\theta} \quad . \tag{50}$$

Equations (31) and (32) become

$$\cosh\beta = \frac{ct}{r} \tag{51}$$

$$\sinh\beta = -[(\frac{ct}{r})^2 - 1]^{\frac{1}{2}} \quad . \tag{52}$$

Thus for $r \le ct$:

$$\tau_{\theta z} = \frac{c}{\pi}\frac{(\tau_1 + \tau_2)(\sin\theta)[(ct)^2 - r^2]^{\frac{1}{2}}}{[(ct)^2 - r^2\sin^2\theta]} \quad . \tag{53}$$

The cylindrical wave vanishes, of course, altogether for $\tau_1 = -\tau_2$. For $r/ct << 1$ we recover the behavior shown in Eq. (39) for $\varkappa = 1$.

For step surface tractions the integration of (36) yields

$$I_{\theta z} = \frac{1}{2}\frac{c}{\pi r}(T_1+T_2)\int_{br}^{t}\left\{\frac{(s^2-b^2r^2)^{\frac{1}{2}}}{s-br\ \sin\theta} - \frac{(s^2-b^2r^2)^{\frac{1}{2}}}{s+br\sin\theta}\right\} ds \tag{54}$$

where we have introduced the slowness b ,

$$b = \frac{1}{c} \quad . \tag{55}$$

This integral can be evaluated to yield for $br \le t$

$$I_{\theta z} = \frac{T_1+T_2}{\pi} \left\{ \sin\theta\ \ell n\left[\frac{(t^2-b^2r^2)^{\frac{1}{2}}+t}{br}\right] - \frac{1}{2}\cos\theta \sin^{-1}\left[\frac{t\sin\theta - br}{t - br\sin\theta}\right] - \frac{1}{2}\cos\theta\sin^{-1}\left[\frac{t\sin\theta + br}{t+br\sin\theta}\right]\right\} \quad . \tag{56}$$

For $\theta \le \frac{\pi}{2}$ the shear stress is then obtained as

$$\tau_{\theta z} = T_1 \cos\theta\ 1(t-br\sin\theta) + I_{\theta z}\ 1(t-br) \quad . \tag{57}$$

For $\theta \ge \frac{\pi}{2}$ we have

$$\tau_{\theta z} = -\ T_2 \cos\theta\ 1(t-br\sin\theta) + I_{\theta z}\ 1(t-br) \quad . \tag{58}$$

For $br/t \ll 1$ the behavior is as indicated by (49).

The case $\underline{\varkappa = 2}$ corresponds to a semi-infinite slit in an unbounded medium subjected to different shear tractions on the two faces. For

$$\tau_1 = \tau_2 = \tau \quad , \tag{59}$$

Eq. (36) reduces to

$$\tau_{\theta z} = -\frac{c}{r}\frac{2^{\frac{1}{2}}}{\pi}\frac{\tau \sinh \frac{1}{2}\beta \sin \frac{1}{2}\theta (1+\cosh \beta + \cos \theta)}{(\cosh \beta)^2 - (\sin \theta)^2} \quad . \tag{60}$$

For $\varkappa = 2$, Eq. (32) becomes

$$\sinh \tfrac{1}{2}\beta = -2^{-\frac{1}{2}}\left(\frac{ct}{r} - 1\right)^{\frac{1}{2}} \quad . \tag{61}$$

After some further manipulation we then obtain for $r < ct$

$$\tau_{\theta z} = \frac{\tau}{\pi} 2^{-\frac{1}{2}}\left(\frac{ct}{r} - 1\right)^{\frac{1}{2}}\left\{\frac{\cos \frac{1}{2}(\frac{\pi}{2} - \theta)}{t-(r/c)\sin \theta} - \frac{\sin \frac{1}{2}(\frac{\pi}{2} - \theta)}{t+(r/c)\sin \theta}\right\} \quad . \tag{62}$$

The behavior for $r/ct << 1$ agrees with what is obtained from Eq. (39) for $\varkappa = 2$.

The integrals to determine the stress for step surface fractions can again be evaluated explicitly, and we obtain for $br \le t$

$$\left(\frac{2r}{c}\right)^{\frac{1}{2}}\frac{\pi J_{\theta z}}{2T} = \{(t-br)^{\frac{1}{2}} + [br(1-\sin \theta)]^{\frac{1}{2}} \tan^{-1}[\frac{t-br}{br(1-\sin \theta)}]^{\frac{1}{2}}\} \cos \tfrac{1}{2}(\tfrac{\pi}{2} - \theta)$$

$$- \{(t-br)^{\frac{1}{2}} + [br(1+\sin \theta)]^{\frac{1}{2}} \tan^{-1}[\frac{t-br}{br(1+\sin \theta)}]^{\frac{1}{2}}\} \sin \tfrac{1}{2}(\tfrac{\pi}{2} - \theta) \tag{63}$$

where T is the applied surface traction. For $0 \le \theta \le \frac{\pi}{2}$ and $2\pi \ge \theta \ge \frac{3\pi}{2}$ the shear stress $\tau_{\theta z}$ is obtained as

$$\tau_{\theta z} = T \cos \theta \; 1(t - br \sin \theta) + J_{\theta z}\, 1(t-br) \quad . \tag{64}$$

For $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ we find

$$\tau_{\theta z} = J_{\theta z}\, 1(t-br) \quad . \tag{65}$$

If $r/ct << 1$ the singular behavior is of the form

$$\frac{\pi \tau_{\theta z}}{2T} \sim \left(\frac{ct}{2r}\right)^{\frac{1}{2}} \{\cos \tfrac{1}{2}(\tfrac{\pi}{2} - \theta) - \sin \tfrac{1}{2}(\tfrac{\pi}{2} - \theta)\} \quad , \tag{66}$$

which agrees with (48).

Equations (64) and (65) when superposed on a uniform shear stress $\tau_{yz} = - T$ yield the solution for the stress $\tau_{\theta z}$ due to the sudden opening of a semi-infinite crack in an unbounded medium which was in a state of uniform shear prior to fracture. As is common in crack problems we find that the stress is singular near the crack tip as $r^{-\frac{1}{2}}$.

CONCLUDING REMARKS

Although only the case $\varkappa \geq 0.5$ was treated here, it is clear that wave propagation in wedges with interior angles less than $\pi/2$ can be treated by using superposition in conjunction with symmetry or anti-symmetry properties. Thus if the face $\theta = \varkappa^* \pi$ $(0.25 \leq \varkappa^* < 0.5)$ is free of traction, and $\theta = 0$ is subjected to $\tau_1\, \delta(t)$, we can simply use (36) with $\varkappa = 2\varkappa^*$, and $\tau_2 = -\tau_1$ to obtain the solution. If the face $\theta = \varkappa^* \pi$ is clamped, i.e., $w \equiv 0$, we substitute $\varkappa = 2\varkappa^*$ and $\tau_2 = \tau_1$ in Eq. (36). For $\varkappa^{**} < 0.25$ the solution for $0.25 \leq \varkappa^* < 0.5$ has to be used in the just described procedure.

It is finally concluded that the method of homogeneous functions in an efficient method to study the dynamic response of an elastic wedge to spatially uniform shear tractions. For surface tractions varying in time as Heaviside step functions it was found that for interior angles $\varkappa\pi$, the singularity of the shear stress $\tau_{\theta z}$ is of the form $[r^{(1/\varkappa)-1}/(1-\varkappa)]$, which shows that the stress is singular only for $\varkappa \geq 1$.

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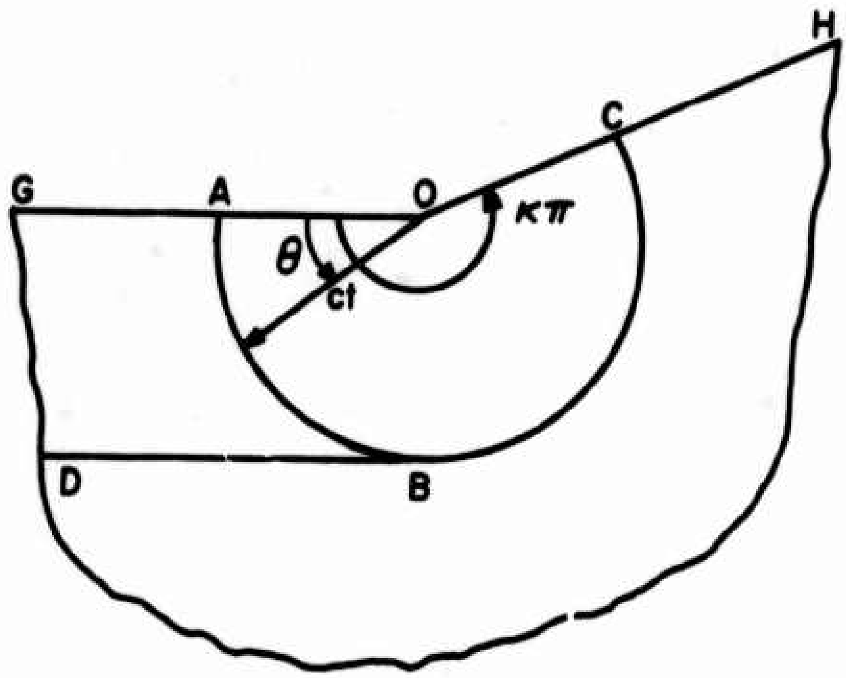

Fig. 1. Wave fronts at time t

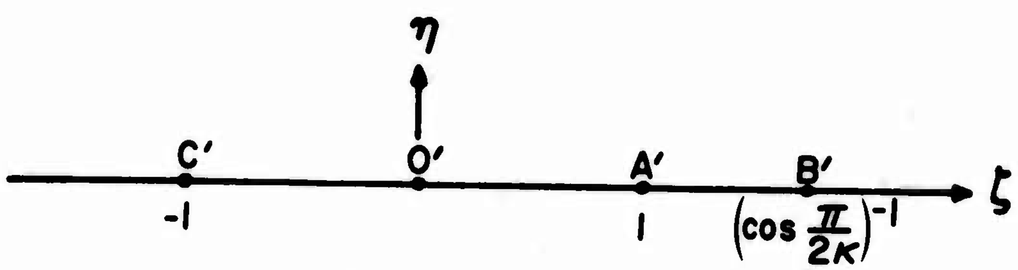

Fig. 2. The ζ-plane for $\kappa > 1$

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13. ABSTRACT

An elastic wedge of interior angle $\chi\pi$ is subjected to spatially uniform but time dependent shear tractions, which are applied to one or both faces of the wedge, parallel to the line of intersection of the faces. The transient wave propagation problem is solved by taking advantage of the dynamic similarity which characterizes problems without a fundamental length in the geometry. The shear stress $\tau_{\theta z}$ is evaluated, and it is found that the singularity hear the vertex of the wedge is of the form $r^{(1/\chi)-1}/(1-\chi)$. The results show that the stress is not singular for interior angles less than π. As a special case we obtain the dynamic shear stress generated by the sudden opening of a semi-infinite crack in a homogeneously sheared unbounded medium.

r to the ((1/x)-1) power

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